IOWA STATE UNIVERSITY
Digital Repository

Retrospective Theses and Dissertations

Iowa State University Capstones, Theses and Dissertations

1945

Transient response of four-terminal networks

Wilton Robert Abbott
Iowa State College

Follow this and additional works at: https://lib.dr.iastate.edu/rtd

Part of the Electrical and Electronics Commons

Recommended Citation
Abbott, Wilton Robert, "Transient response of four-terminal networks " (1945). *Retrospective Theses and Dissertations*. 13551.
https://lib.dr.iastate.edu/rtd/13551

This Dissertation is brought to you for free and open access by the Iowa State University Capstones, Theses and Dissertations at Iowa State University Digital Repository. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.

المنارة للاستشارات

www.manaraa.com

NOTE TO USERS

This reproduction is the best copy available.

UMI®

TRANSIENT RESPONSE OF FOUR-TERMINAL NETWORKS

by

Wilton Robert Abbott

A Thesis Submitted to the Graduate Faculty
for the Degree of

DOCTOR OF PHILOSOPHY

Major Subject: Electrical Engineering

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State College

1945

UMI Number: DP12507

INFORMATION TO USERS

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleed-through, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

UMI®

UMI Microform DP12507

Copyright 2005 by ProQuest Information and Learning Company.

All rights reserved. This microform edition is protected against unauthorized copying under Title 17, United States Code.

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

TK3226
A629t

TABLE OF CONTENTS

I. INTRODUCTION 1

A. Introductory Remarks 1

B. Mathematical Methods 2

C. Scope of the Study 4

D. Laplace Transform 6

II. REVIEW OF LITERATURE 10

III. RESPONSE OF PASSIVE NETWORKS 11

IV. RESPONSE OF LINEAR NETWORKS CONTAINING VACUUM TUBES . 15

V. FOUR-TERMINAL NETWORKS 20

VI. TWO NETWORK THEOREMS 30

A. Superposition Theorem 30

B. Thevenin's Theorem 30

VII. APPLICATION 33

A. Theoretical 33

B. Experimental 38

VIII. CONCLUSIONS 43

IX. LITERATURE CITED 44

X. ACKNOWLEDGMENTS 46

XI. VITA 47

XII. APPENDICES 49

A. Matrix Algebra 49

B. Circuit Analysis 54

77974

C. Symbols 58

D. Tabulated Data 59

LIST OF ILLUSTRATIONS

1. Four Terminal Network -- Current and Voltage Conventions 21
2. Simple Vacuum Tube Amplifier 21
3. Vacuum Tube Amplifier 21
4. Shunt Peaked Wide Band Amplifier 26
5. Output Voltage with Unit Step Applied 26
6. Circuit Used in Experiment 34
7. Arrangement of Equipment in Laboratory 39
8. Oscillogram of Signal Voltage 42
9. Oscillogram of Output Voltage Without Feedback . . . 42
10. Oscillogram of Output Voltage With Feedback 42

I. INTRODUCTION

A. Introductory Remarks

A system changing from one steady state to another is in a transient condition. For instance, a tree having been cut by a lumberman is in a transient condition of motion while falling from the steady state of a living perpendicular to that of a dead horizontal.

The study of transients has been more systematized in Electrical Engineering perhaps than in any other branch of learning, but it is important in many other branches. For instance History, Sociology, Biology, and Geology among others find in many of their problems the change from one steady state to another.

If any justification is necessary for the study of transients in Electrical Engineering, it may be found in the following quotation from the Bell System Technical Journal (8, p. 139).

> Since all voice and sound transmission, particularly that of orchestral music, may be regarded as essentially a transient problem, it is appropriate that we visualize the effects on the complex wave forms of any distortions which may be present in the transmission apparatus.

Going back as it does into the last century, the study of electrical transients is not new. In fact, given sufficient

information about a linear circuit and sufficient time, patience, and mathematical skill it is possible to carry through the complete transient analysis of any such circuit. Certain special non-linear circuits can be analyzed exactly, and approximate numerical solutions of many simple non-linear circuits can be found. However, for a literal solution of general non-linear circuits the engineer must wait until the mathematician has found a method of solving literal non-linear differential equations. In the following, linear circuits will be considered exclusively.

B. Mathematical Methods

Although there are several methods available for solving circuits, the method of the Laplace Transform has been chosen for the problems to be considered here. While each method of attack has types of problems for which it is the best suited, the Laplace Transform is believed to be the best for more types than any of the others. Gardner and Barnes (3) is probably the best textbook available using the Laplace Transform. For a complete exposition of this powerful method the reader is referred to that book. However a brief review of the subject is included.

The Laplace Transform is the only type of transform to be used here. Consequently no ambiguity is introduced if the term "transform of the current, voltage, etc." is used to mean

"Laplace Transform of the current, voltage, etc.". It is to be understood that lower case letters representing dependent functions of time will refer to the functions themselves, while upper case letters will represent the corresponding transforms. Wherever no ambiguity results, indication of the independent variable will be omitted with the understanding that t is the independent variable for the time functions and s for the transforms.

The term "response" should be defined at this point. A network is assumed to be excited by one or more current or voltage sources. Any current or voltage of interest which results from the excitation is the "response".

In the work to follow the transform of the response will usually be considered to be the solution of the problem. Finding the time function corresponding to the response transform will then be strictly mathematical in character. This work is adequately covered in books on transform theory.

Work with the transforms of linear differential equations involves a considerable amount of manipulation of systems of linear algebraic equations. These are handled most readily by means of matrix algebra. This algebra is described briefly in Guillemin (4) and more completely in Bôcher (2). It will also be reviewed briefly in the appendix.

C. Scope of the Study

The body of the thesis can be divided into three parts. In sections three and four some general relationships between the transient and a-c responses of both passive networks and networks containing vacuum tubes are developed. These relationships are stated in Theorems I and II. Theorem I is apparently not new. Its subject matter is hinted at and the results made use of by several authors. However the author of this thesis has never seen the principle itself or the conditions governing its use clearly stated. Therefore it is felt that it should be included.

In section five the general theory of the transient response of four-terminal networks is developed. Previous work in this field has been restricted to linear, passive, bilateral networks. Section five extends this work to include networks containing linear (class A operated) vacuum tubes. It is shown there that such networks are equivalent to linear, passive, unilateral networks. The fact that the network is simultaneously linear and unilateral implies that energy is not conserved as far as the two pairs of terminals are concerned. This is possible because the network is not really passive although it may be represented as such.

An actual problem is set up and solved in section six, the results being verified experimentally. It should not be inferred that this is an attempt at experimental verification of

the theory developed here. If a mathematical development is based on consistent fundamental premises, and if the manipulations of the development are made without error, then the results will be consistent with the fundamental premises. In terms of physical theory, if certain basic principles are known to be true, and if all the factors relevant to these principles are included in a development using mathematics known to apply to the situation, then the results of the development will be true. This principle cannot be applied generally because even if the basic principles are known, inclusion of all the factors would make the problem so complicated, in many cases, that the mathematics would be impossible to handle. Workers in linear circuit theory are fortunate because: (a) certain basic principles are known to be true from thousands of experiments, (b) often all the factors can be included without unduly complicating the mathematics, and (c) the mathematics is usually straightforward and relatively simple to apply. Thus if experimentation in this field fails to agree with theory which has been developed carefully, the error must of necessity be ascribed to the experimenter and not to the theory. For this reason the experimental work described herein is in the nature of illustration rather than of verification.

D. Laplace Transform

The following is a brief discussion of the Laplace Transform method of solving differential equations. It is intended to serve both as a brief review and to define symbols.

Consider a function of time $f(t)$ which meets certain conditions as to transformability (3, p. 102). These conditions are met by most functions which occur in work with electrical circuits. By definition the following relations hold,

$$(1\text{-}1) \quad F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)\, \varepsilon^{-st} dt$$

The symbol $\mathcal{L}$ is to be read, "Laplace Transform of". If the function is transformable, the integral will converge.

By consideration of the properties of the integral it is evident that the transform of a constant times a function equals the constant times the transform of the function. Also the transform of the sum of functions equals the sum of the transforms of the functions. Thus the Laplace Transformation is a linear operation.

Integrating by parts it is easily seen that

$$(1\text{-}2) \quad \mathcal{L}\left[\frac{d(f(t))}{dt}\right] = sF(s) - f(0+)$$

and

(1-3) $\mathcal{L}\left[\int_0^t f(t)\,d\tau\right] = \frac{F(s)}{s}$

Higher order derivatives and integrals (although they need not appear in circuit theory) are also in the form of simple algebraic functions when transformed.

The transforms of the time functions usually appearing in circuit theory are also simple algebraic functions. The most important of these is the exponential function.

(1-4) $\mathcal{L}\left[\varepsilon^{-at}\right] = \frac{1}{s+a}$

Closely related to the exponential function are the constant, the sine, and the cosine function.

(1-5) $\mathcal{L}\left[1\right] = \frac{1}{s}$

(1-6) $\mathcal{L}\left[\sin\beta t\right] = \frac{\beta}{s^2+\beta^2}$

(1-7) $\mathcal{L}\left[\cos\beta t\right] = \frac{s}{s^2+\beta^2}$

$$(1\text{-}8) \quad 2\mathcal{L}\left[\varepsilon^{-\alpha t}\cos(\beta t+\theta)\right] = \frac{\varepsilon^{j\theta}}{s+\alpha-j\beta} + \frac{\varepsilon^{-j\theta}}{s+\alpha+j\beta}$$

The direct transform is usually easy to find by use of equation 1-1. The inverse problem, that of finding $f(t)$ given $F(s)$, is usually quite difficult. An explicit solution to this problem is known but should be avoided wherever possible. A much more satisfactory procedure is to use a table of transforms which has been prepared by transforming a large number of functions. Such a table is given in Gardner & Barnes (3, p. 334-356). If such a table is not readily available, most transforms arising from circuit problems can be inverted by expanding in partial fractions and making use of 1-4 and 1-5. The case of multiple poles requires special treatment which will not be given here but may be found in reference (3, p. 159).

As an example consider the inversion of 7-8.

$$(1\text{-}9) \quad F(s) = \frac{s+a_0}{(s+\alpha)^2+\gamma^2} = \frac{s+a_0}{(s+\alpha-j\gamma)(s+\alpha+j\gamma)}$$

Expanding in partial fractions,

$$(1\text{-}10) \quad F(s) = \frac{a_0-\alpha+j\gamma}{j2\gamma(s+\alpha-j\gamma)} + \frac{a_0-\alpha-j\gamma}{-j2\gamma(s+\alpha+j\gamma)}$$

(1-11) $$F(s) = \frac{\frac{1}{2} - j\,\frac{a_0 - \alpha}{2\gamma}}{s + \alpha - j\gamma} + \frac{\frac{1}{2} + j\,\frac{a_0 - \alpha}{2\gamma}}{s + \alpha + j\gamma}$$

(1-12) $$F(s) = \frac{1}{2}\sqrt{1 + \frac{(a_0 - \alpha)^2}{\gamma^2}}\left[\frac{\varepsilon^{j\theta}}{s + \alpha - j\gamma} + \frac{\varepsilon^{-j\theta}}{s + \alpha + j\gamma}\right]$$

where

(1-13) $$\theta = \tan^{-1}\frac{\alpha - a_0}{\gamma}$$

From 1-8

(1-14) $$f(t) = \sqrt{1 + \frac{(a_0 - \alpha)^2}{\gamma^2}}\;\varepsilon^{-\alpha t}\cos(\gamma t + \theta)$$

Application of the Laplace Transform to circuit problems will be discussed in appendix B.

II. REVIEW OF LITERATURE

Published material on the subject of this thesis is very sparse. There has apparently been no attempt to systematize the problem except for the material reviewed in Guillemin (4, p. 474-506). Unfortunately the method of attack given there yields usable results only if the system is highly idealized. Other references treat each type of four-terminal network as a new problem rather than trying to find the basic principles which bind them all together.

When it comes to a consideration of networks containing class A operated vacuum tubes, the situation is even worse. There has apparently been no attempt to systematize the problem. References (1) and (5) are typical of those examined. Reference (1) treats a particular amplifier using the Fourier Series as a method of analysis. Reference (5) treats the basic audio frequency amplifiers using the Cauchy-Heaviside Operational Calculus. Each amplifier is treated as a new problem.

On the other hand, the general subject of transients is covered in a large number of references. Gardner and Barnes (3, p. 367-382) contains a bibliography of 364 items. Reference is made to that bibliography for the use of anyone who wants an extensive listing.

III. RESPONSE OF PASSIVE NETWORKS

Consider an (n + 1)-node linear network. In general each node will be coupled to each other node as discussed in appendix B. It will be assumed that an a-c current source is connected from node 0 to 1 and that the network is otherwise passive. The steady state a-c response is desired.

The total current leaving node 1 is given by,

$$(3\text{-}1) \quad I(j\omega) = V_1(j\omega)\left[G_{11} + j\omega C_{11} + \frac{\Gamma_{11}}{j\omega}\right] + V_2(j\omega)\left[G_{12} + j\omega C_{12} + \frac{\Gamma_{12}}{j\omega}\right] + \cdots$$

Where $j = \sqrt{-1}$, $\omega = 2\pi$ times the frequency, I is the output of the current source, the V's are the various node potentials, and the G's, C's, and Γ's are defined in appendix B.

There will be a similar equation for each node except for the reference node. The entire set can be written as the matrix equation 3-2.

$$(3\text{-}2) \quad \left[\left\| G_{k\ell} \right\| + j\omega \left\| C_{k\ell} \right\| + \frac{1}{j\omega} \left\| \Gamma_{k\ell} \right\|\right] \times \left\| V_k(j\omega) \right\| = \left\| I_k(j\omega) \right\| \quad , \quad k, \ell = 1, n$$

Where $I_k = I$ when $K = 1$ and $I_k = 0$ when $K \neq 1$.

In appendix B a similar equation is developed for the transient response of an n-node network. Equation 3-3 is identical with 12-21.

$$(3\text{-}3) \quad \left[\left\|G_{k\ell}\right\| + s\left\|C_{k\ell}\right\| + \frac{1}{s}\left\|\Gamma_{k\ell}\right\|\right] \times \left\|V_k(s)\right\| = \left\|I_k(s)\right\| \quad , \quad k, \ell = 1, n$$

if again $I_K = I$ when $K = 1$ and $I_K = 0$ when $K \neq 1$. This will be true if there is no initial energy storage in the network and if the only driving source is connected from node 0 to 1.

Comparing 3-2 with 3-3 it is apparent that both can be written

$$(3\text{-}4) \quad \left\|Y_{k\ell}(\lambda)\right\| \times \left\|V_k(\lambda)\right\| = \left\|I_k(\lambda)\right\| \quad , \quad k, \ell = 1, n$$

where λ may be either s or $j\omega$.

If, by definition,

$$(3\text{-}5) \quad \left\|Y_{k\ell}(\lambda)\right\|^{-1} = \left\|Z_{k\ell}(\lambda)\right\|$$

$$(3\text{-}6) \quad \left\|V_k(\lambda)\right\| = \left\|Z_{k\ell}(\lambda)\right\| \times \left\|I_k(\lambda)\right\|$$

Expanding this matrix equation yields,

$$(3\text{-}7) \quad V_k(\lambda) = Z_{k1}(\lambda)\, I(\lambda)$$

Therefore the potential of any node (either the a-c potential or the transform of the potential) equals a function of λ times

the output of the source (either the a-c current or the current transform).

The voltage between nodes b and c is then

$$(3\text{-}8) \quad V_b(\lambda) - V_c(\lambda) = [Z_{b1}(\lambda) - Z_{c1}(\lambda)]\, I(\lambda)$$

If the admittance between these nodes is Y_{bc} and the current is I_{bc}, then

$$(3\text{-}9) \quad I_{bc}(\lambda) = Y_{bc}(\lambda)[Z_{b1}(\lambda) - Z_{c1}(\lambda)]\, I(\lambda)$$

Equation 3-8 or 3-9 gives the response of the network. The fact that in all of this λ may be either s or $j\omega$ makes it possible to state Theorem I of which the above is a proof. (It should be noted that by an exchange of sources the excitation may be either a voltage or a current.)

THEOREM I. If the ratio of the a-c response to the excitation in a linear passive network is known as a function of $j\omega$, then the ratio of the response transform to the excitation transform is found by substituting s for $j\omega$ whenever there is no initial energy storage in the network. The converse is also true.

In the development of the Laplace Transform theory it is shown that s is essentially a complex variable that is, $s = \sigma + j\omega$. From the point of view of Function Theory the a-c response is a special case of the transient response where the a-c behavior is the behavior along the axis of imaginaries.

This theorem will be of considerable use in the study of the transient response of four-terminal networks as these systems are well understood in the steady state. It will also be of use as engineers are usually skilled in setting up a-c relationships.

IV. RESPONSE OF LINEAR NETWORKS CONTAINING VACUUM TUBES

The results of the preceding section apply particularly to linear, passive networks. This completely excludes the use of vacuum tube amplifiers; thus a large class of useful networks is not considered. These results will now be extended to include the case of networks containing class A amplifiers. By restricting the amplifiers to class A operation the circuit is maintained linear, and the conventional equivalent circuit may be used. This extended class of networks will hereafter be denoted as "class A" networks.

It will be instructive to show a derivation of the equivalent circuit of the class A operated vacuum tube and to show that it is applicable to transient as well as sinusoidal voltages.

The plate current of a vacuum tube may be represented by a power series as is shown in textbooks of electronics. Strict class A operation implies that all terms of degree higher than one are zero. Using primes to denote steady values and unprimed symbols to denote variable values,

$$(4\text{-}1) \quad i_p + i_p' = k_1(\mu e_c + e_p) + k_1(\mu e_c' + e_p') + f(e_c', e_p').$$

Where μ is the amplification factor of the tube, e_c is the

potential of the grid with respect to the cathode, e_p is the potential of the plate with respect to the cathode, k_1 is a constant, and $f(e_c', e_p')$ denotes higher degree terms in the steady voltages. (These higher degree terms are present because curvature of the tube characteristics is significant for the d-c voltages even though it is negligible for the varying voltages.) By the Principle of Superposition the varying current is a result only of the varying voltage. Therefore

(4-2) $$i_p = k_1(\mu e_c + e_p)$$

From this

(4-3) $$\frac{e_p}{i_p} = \frac{1}{k_1} - \frac{\mu e_c}{i_p}$$

This may be interpreted as a resistance, or rather, two resistances in series. The first is $1/k_1$ and the second is $(-\mu e_c/i_p)$. By the Compensation Theorem the varying resistance $(-\mu e_c/i_p)$ may be replaced by a generator $+\mu e_c$ with its voltage rising from plate to cathode. Define the first resistance as follows

(4-4) $$\frac{1}{k_1} = r_p$$

This is called the plate resistance. Since no circuit elements

but generators and resistors are involved and no restrictions were placed on e_c, the representation applies to transient analyses.

In many cases an exchange of sources is desirable. In this case the tube is equivalent to a current generator of output $\mu e_c/r_p = g_m e_c$ flowing from plate to cathode paralleled by a resistance r_p. The quantity g_m is the transconductance of the tube.

An $(n+1)$-node class A network containing two tubes will be assumed. Generalization to any number of tubes will be immediately evident. There will be no great loss in generality if both cathodes are at reference potential. Take the grid of the first tube at node 1 and its anode at node 2. Let the grid of the other tube be at node $(n-1)$ and its anode be at node n. Assume both have the same transconductance, g_m. Let the driving source be a current generator from the reference node to node 1. Then the transform equation of the network is

$$(4\text{-}5) \qquad \| Y_{k\ell} \| \times \| V_k \| = \left\| \begin{matrix} I \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{matrix} \right\| + \left\| \begin{matrix} 0 \\ -g_m V_1 \\ 0 \\ \vdots \\ 0 \\ -g_m V_{n-1} \end{matrix} \right\| , \quad k, \ell = 1, n$$

$$(4\text{-}6) \qquad \| Y_{k\ell} \| \times \| V_k \| = \left\| \begin{matrix} I \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{matrix} \right\| + \left\| \begin{matrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -g_m & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & -g_m & 0 \end{matrix} \right\| \times \| V_k \| , \quad k, \ell = 1, n$$

$$(4\text{-}7)\qquad \left[\left\| Y_{k\ell} \right\| + \left\| \begin{matrix} 0 & 0 & \cdot & \cdot & 0 & 0 \\ +g_m & 0 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & +g_m & 0 \end{matrix} \right\|\right] \times \left\| V_k \right\| = \left\| \begin{matrix} I \\ 0 \\ \cdot \\ \cdot \\ 0 \end{matrix} \right\| \qquad k, \ell = 1, n$$

This equation is of the form

$$(4\text{-}8)\qquad \left\| \bar{Y}_{k\ell} \right\| \times \left\| V_k \right\| = \left\| I_k \right\|$$

where

$$(4\text{-}9)\qquad \left\| \bar{Y}_{k\ell} \right\| = s \left\| C_{k\ell} \right\| + \frac{1}{s} \left\| \Gamma_{k\ell} \right\| + \left\| G_{k\ell} \right\| + \left\| \begin{matrix} 0 & 0 & \cdot & \cdot & 0 & 0 \\ +g_m & 0 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & +g_m & 0 \end{matrix} \right\|$$

Adding more tubes and separating the cathodes will merely add more elements to the matrix containing the g_m's. Evidently the effect of adding vacuum tubes is merely to alter the conductance matrix in the admittance making it unsymmetrical. This asymmetry is characteristic of unilateral networks. The equation is still linear.

The only difference between equations 3-3 and 4-8 is in the asymmetry of admittance matrix in 4-8. If the a-c response were to be calculated, the only difference would be that $j\omega$ would be substituted for s in 4-9. This follows from the same reasoning as in section III since the s or $j\omega$ terms occur only with the inductances and capacitances.

The proof of Theorem I did not require the symmetry of the admittance matrix. Thus Theorem II is proved.

THEOREM II. If the ratio of the a-c response to the excitation in a class A network is known as a function of $j\omega$, then the ratio of the response transform to the excitation transform is found by substituting s for $j\omega$ whenever there is no initial energy storage in the network. The converse is also true.

In the succeeding pages certain methods of solving network problems will be developed. It should not be inferred that these are the only methods to which Theorems I and II apply, or even that they will be the best methods to use in any particular case. Theorems I and II give the relationship of the transient response to the a-c response when the a-c response is found by any means which may be convenient.

V. FOUR-TERMINAL NETWORKS

In reference (4, p. 132-180) Guillemin develops in great detail the characteristics of linear, passive, bilateral, four-terminal networks. This section extends a portion of his work to class A, four-terminal networks. In comparing the material of this section with Guillemin's work it should be remembered that in bilateral networks the Y and Z matrices are symmetrical. This symmetry does not exist generally in class A networks. Because of the symmetry Guillemin occasionally simplifies his expressions by reversing the order of subscripts. This is not permissible in class A networks.

Making use of Theorem II the indication of the independent variable will be omitted in the following work. It will be understood that if the a-c response is desired, the variable will be $j\omega$; and if the transient response is desired, the variable will be s. There will, of course, be no initial energy storage in the network when dealing with the transient response.

The conventions as to current and voltage are shown on figure 1. Any of the four variables I_1, I_2, V_1, or V_2 may be the excitation with any one of the remaining three the response.

There are six ways in which these variables can be related. The six ways are interdependent since the four quantities

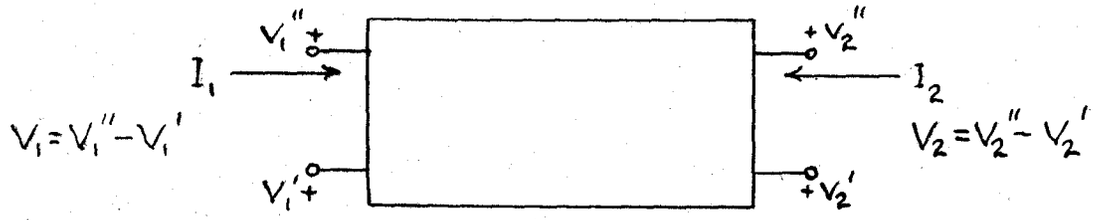
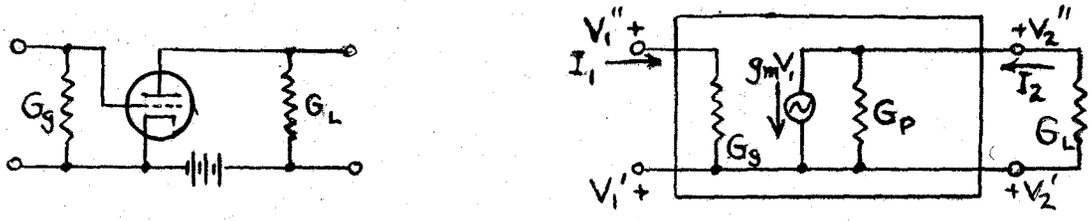
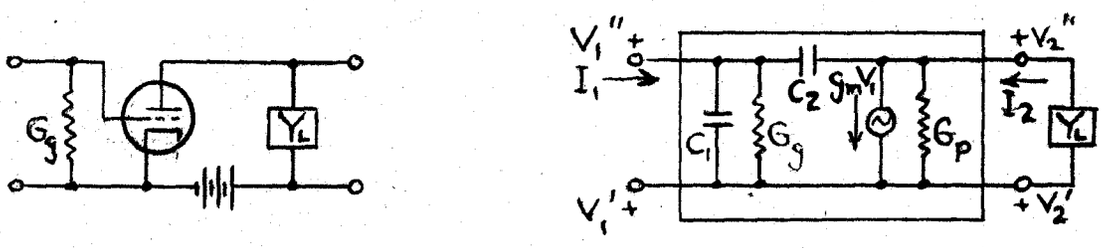


FIG. 1. FOUR-TERMINAL NETWORK — CURRENT AND VOLTAGE CONVENTIONS



A. ACTUAL CIRCUIT B. EQUIVALENT CIRCUIT

FIG. 2. SIMPLE VACUUM TUBE AMPLIFIER



A. ACTUAL CIRCUIT B. EQUIVALENT CIRCUIT

FIG. 3. VACUUM TUBE AMPLIFIER

themselves are not affected by the method chosen to represent them. The six matrix equations are written below using Guillemin's notation.

(5-1a) $$\begin{Vmatrix} I_1 \\ I_2 \end{Vmatrix} = \begin{Vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{Vmatrix} \times \begin{Vmatrix} V_1 \\ V_2 \end{Vmatrix}$$

(5-1b) $$\begin{Vmatrix} V_1 \\ V_2 \end{Vmatrix} = \begin{Vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{Vmatrix} \times \begin{Vmatrix} I_1 \\ I_2 \end{Vmatrix}$$

(5-2a) $$\begin{Vmatrix} I_1 \\ V_2 \end{Vmatrix} = \begin{Vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{Vmatrix} \times \begin{Vmatrix} V_1 \\ I_2 \end{Vmatrix}$$

(5-2b) $$\begin{Vmatrix} V_1 \\ I_2 \end{Vmatrix} = \begin{Vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{Vmatrix} \times \begin{Vmatrix} I_1 \\ V_2 \end{Vmatrix}$$

(5-3a) $$\begin{Vmatrix} V_1 \\ I_1 \end{Vmatrix} = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} \times \begin{Vmatrix} V_2 \\ -I_2 \end{Vmatrix}$$

(5-3b) $$\begin{Vmatrix} V_2 \\ -I_2 \end{Vmatrix} = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix}^{-1} \times \begin{Vmatrix} V_1 \\ I_1 \end{Vmatrix}$$

The relationships among the various coefficients and methods of calculating them for an actual network will now be found. Since the paired relationships 5-1, 5-2, and 5-3

are reciprocals, it will be necessary to calculate only the "a" member of each pair.

By inspection of 5-1a it is seen that y_{11} is the short circuit input admittance at terminal pair 1 (to short circuit the output, set $V_2 = 0$), that y_{21} is the short circuit transfer admittance at terminal pair 1, that y_{22} is the short circuit input admittance at terminal pair 2, and that y_{12} is the short circuit transfer admittance at terminal pair 2. Thus the y's can be determined from the given circuit. The g's and the general circuit constants (A, B, C, and D) will now be calculated from the y's.

Equation 5-1a can be written, performing indicated operations

$$\left\| \begin{matrix} I_1 \\ 0 \end{matrix} \right\| + \left\| \begin{matrix} 0 \\ I_2 \end{matrix} \right\| = \left\| \begin{matrix} y_{11} \\ y_{21} \end{matrix} \right\| V_1 + \left\| \begin{matrix} y_{12} \\ y_{22} \end{matrix} \right\| V_2 \tag{5-4}$$

Rearranging terms and combining gives,

$$\left\| \begin{matrix} -y_{11}V_1 + I_1 \\ -y_{21}V_1 + 0 \end{matrix} \right\| = \left\| \begin{matrix} y_{12}V_2 + 0 \\ y_{22}V_2 - I_2 \end{matrix} \right\| \tag{5-5}$$

This is equivalent to,

$$\left\| \begin{matrix} -y_{11} & 1 \\ -y_{21} & 0 \end{matrix} \right\| \times \left\| \begin{matrix} V_1 \\ I_1 \end{matrix} \right\| = \left\| \begin{matrix} y_{12} & 0 \\ y_{22} & 1 \end{matrix} \right\| \left\| \begin{matrix} V_2 \\ -I_2 \end{matrix} \right\| \tag{5-6}$$

Then

$$(5\text{-}7) \qquad \begin{Vmatrix} V_1 \\ I_1 \end{Vmatrix} = \frac{1}{y_{21}} \begin{Vmatrix} 0 & -1 \\ y_{21} & -y_{11} \end{Vmatrix} \times \begin{Vmatrix} y_{12} & 0 \\ y_{22} & 1 \end{Vmatrix} \times \begin{Vmatrix} V_2 \\ -I_2 \end{Vmatrix}$$

and

$$(5\text{-}8) \qquad \begin{Vmatrix} V_1 \\ I_1 \end{Vmatrix} = \frac{1}{y_{21}} \begin{Vmatrix} -y_{22} & -1 \\ y_{21}y_{12} - y_{11}y_{22} & -y_{11} \end{Vmatrix} \times \begin{Vmatrix} V_2 \\ -I_2 \end{Vmatrix}$$

Comparing 5-8 with 5-3a it is seen that

$$(5\text{-}9) \qquad \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} = -\frac{1}{y_{21}} \begin{Vmatrix} y_{22} & 1 \\ y_{11}y_{22} - y_{21}y_{12} & y_{11} \end{Vmatrix}$$

In a similar manner it is found that

$$(5\text{-}10) \qquad \begin{Vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{Vmatrix} = \frac{1}{y_{22}} \begin{Vmatrix} y_{11}y_{22} - y_{21}y_{12} & y_{12} \\ -y_{21} & 1 \end{Vmatrix}$$

The particular member of the set 5-1a to 5-3a inclusive that should be used depends on the results which are needed. Of course, any one will give all required information, but experience will indicate which is most convenient for a particular case.

To illustrate consider the network shown in figure 2. Let G_g = 2 micromhos, G_p = 1 micromho, G_L = 10 micromhos, and g_m = 1000 micromhos. Then by inspection

(5-11)
$$\begin{cases} y_{11} = 2\times10^{-6} \\ y_{12} = 0 \\ y_{21} = 1000\times10^{-6} \\ y_{22} = 10^{-6} \\ I_2 = -10\times10^{-6}V_2 \end{cases}$$

From these values, the following equations are obtained.

(5-12a)
$$\begin{Vmatrix} I_1 \\ -10^{-5}V_2 \end{Vmatrix} = 10^{-6}\begin{Vmatrix} 2 & 0 \\ 1000 & 1 \end{Vmatrix} \times \begin{Vmatrix} V_1 \\ V_2 \end{Vmatrix}$$

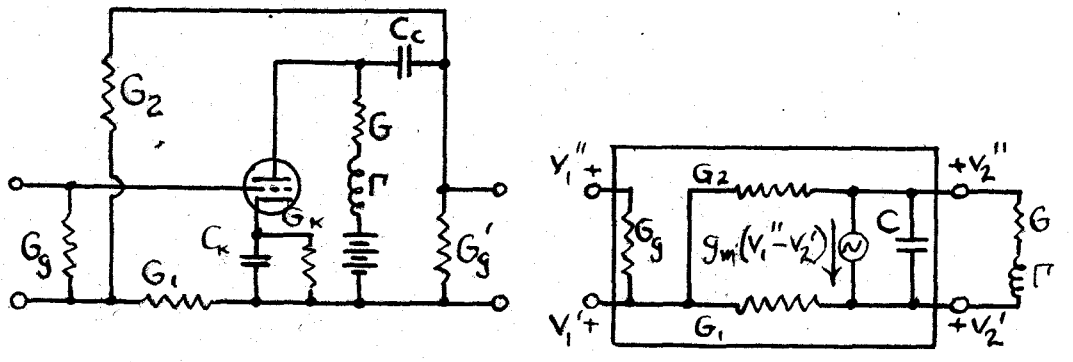
(5-12b)
$$\begin{Vmatrix} V_1 \\ V_2 \end{Vmatrix} = 10^{6}\begin{Vmatrix} 0.5 & 0 \\ -500 & 1 \end{Vmatrix} \times \begin{Vmatrix} I_1 \\ -10^{-5}V_2 \end{Vmatrix}$$

(5-13a)
$$\begin{Vmatrix} I_1 \\ V_2 \end{Vmatrix} = \begin{Vmatrix} 2\times10^{-6} & 0 \\ -1000 & 10^{6} \end{Vmatrix} \times \begin{Vmatrix} V_1 \\ -10^{-5}V_2 \end{Vmatrix}$$

(5-13b)
$$\begin{Vmatrix} V_1 \\ -10^{-5}V_2 \end{Vmatrix} = \frac{1}{2}\begin{Vmatrix} 10^{6} & 0 \\ 1000 & 2\times10^{-6} \end{Vmatrix} \times \begin{Vmatrix} I_1 \\ V_2 \end{Vmatrix}$$

(5-14a)
$$\begin{Vmatrix} V_1 \\ I_1 \end{Vmatrix} = -\begin{Vmatrix} 10^{-3} & 10^{3} \\ 2\times10^{-9} & 2\times10^{-3} \end{Vmatrix} \times \begin{Vmatrix} V_2 \\ 10^{-5}V_2 \end{Vmatrix}$$

(5-14b)
$$\begin{Vmatrix} V_2 \\ 10^{-5}V_2 \end{Vmatrix} = -\frac{1}{0}\begin{Vmatrix} 2\times10^{-3} & -10^{3} \\ -2\times10^{-9} & 10^{-3} \end{Vmatrix} \times \begin{Vmatrix} V_1 \\ I_1 \end{Vmatrix}$$



A. ACTUAL CIRCUIT B. EQUIVALENT CIRCUIT

FIG. 4. SHUNT PEAKED WIDE BAND AMPLIFIER

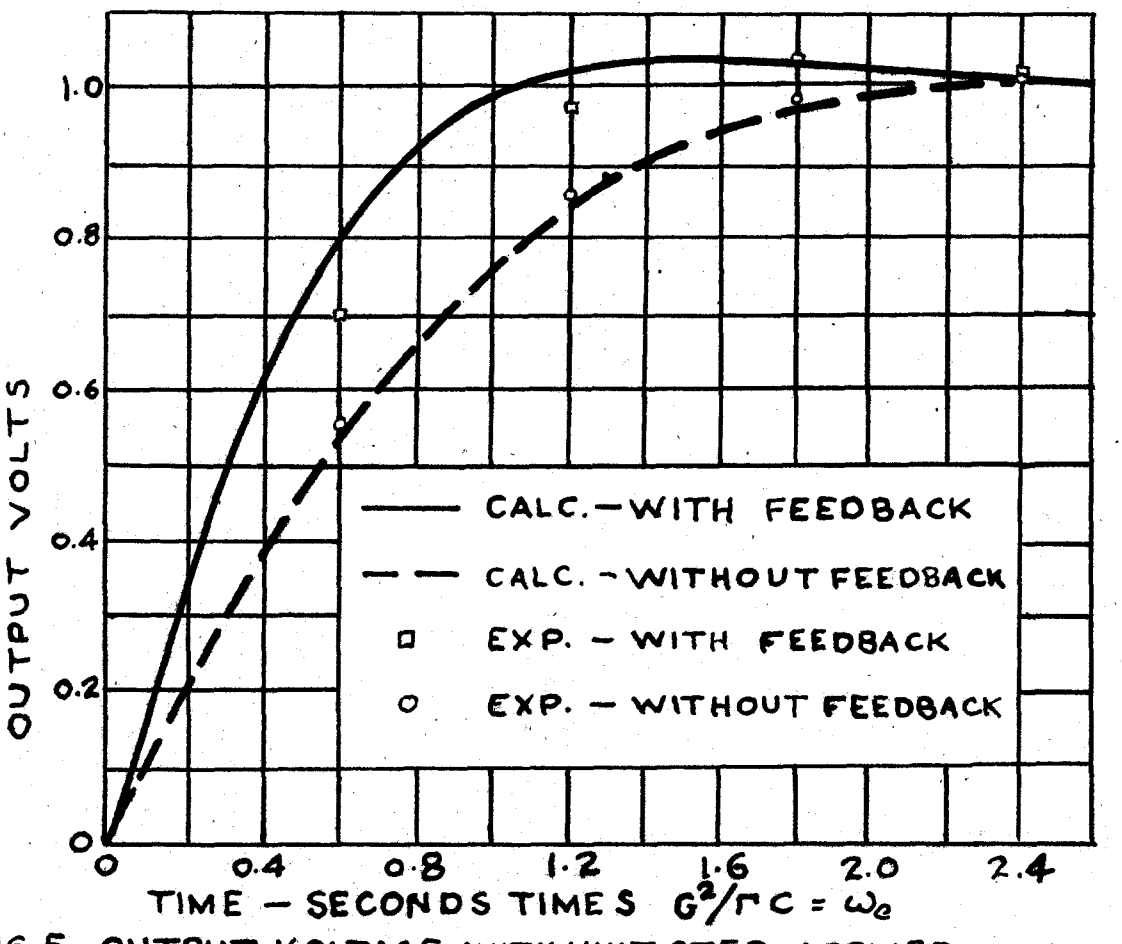


FIG. 5. OUTPUT VOLTAGE WITH UNIT STEP APPLIED (CIRCUITS OF FIG. 4 AND 6)

Equation 5-14b is of indeterminate form, and will not be used. The one of the five remaining equations to be used depends on the results desired. If, for example the amplifier gain and input admittance are desired, equation 5-14a will be most useful. (For the transient point of view the gain is here defined as the ratio of the output voltage transform to the input voltage transform.)

Dividing both sides of 5-14a by V_2 and performing the indicated multiplication gives

$$(5\text{-}15) \qquad \frac{1}{V_2}\left\| \begin{matrix} V_1 \\ I_1 \end{matrix} \right\| = -\left\| \begin{matrix} 10^{-3}+10^{-2} \\ 2\times10^{-9}+2\times10^{-8} \end{matrix} \right\| = -0.011 \left\| \begin{matrix} 1 \\ 2\times10^{-6} \end{matrix} \right\|$$

$$(5\text{-}16) \qquad A' = \frac{V_2}{V_1} = -\frac{1}{.011} = -90.9$$

$$(5\text{-}17) \qquad \frac{V_2}{I_1} = -\frac{10^8}{2.2} = -4.55\times10^7$$

$$(5\text{-}18) \qquad Y_{IN} = \frac{I_1}{V_1} = 2\times10^{-6} \text{ MHOS}$$

Thus important characteristics of the amplifier are readily determined. It will be recognized that this simple example could have been solved more easily by conventional means. However, this is the usual situation when a general method is illustrated by a simple example.

Since a large number of problems involve the calculation of the gain and input admittance of amplifiers, equation 5-3a will be extended a little further. Let the load admittance be Y_L, and connect it to terminal pair 2. Then $I_2 = -Y_L V_2$.

(5-19) $$\left\| \begin{matrix} V_1 \\ I_1 \end{matrix} \right\| = V_2 \left\| \begin{matrix} A & B \\ C & D \end{matrix} \right\| \times \left\| \begin{matrix} 1 \\ Y_L \end{matrix} \right\| = V_2 \left\| \begin{matrix} A + BY_L \\ C + DY_L \end{matrix} \right\|$$

(5-20) $$A' = \frac{V_2}{V_1} = \frac{1}{A + BY_L}$$

(5-21) $$Y_{IN} = \frac{I_1}{V_1} = \frac{C + DY_L}{A + BY_L}$$

In terms of the y's,

(5-20a) $$A' = -\frac{y_{21}}{y_{22} + Y_L}$$

(5-21a) $$Y_{IN} = y_{11} + A' y_{12}$$

These results will be applied to another example. This circuit is shown on figure 3. It is the same as figure 2 but with the interelectrode capacitances taken into consideration. The plate-cathode capacitance is included in Y_L. Substitution of s for $j\omega$ in the a-c admittances will be made directly.

(5-22) $$\begin{Vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{Vmatrix} = \begin{Vmatrix} G_g + s(C_1 + C_2) & -sC_2 \\ g_m - sC_2 & G_p + sC_2 \end{Vmatrix}$$

Then

(5-23) $$A' = \frac{V_2(s)}{V_1(s)} = -\frac{g_m - sC_2}{G_p + Y_L + sC_2}$$

(5-24) $$Y_{IN} = G_g + sC_1 + sC_2(1 - A')$$

VI. TWO NETWORK THEOREMS

A. Superposition Theorem

Equation 4-8 can be extended to include the case where there is more than one independent current source. In this case

$$\left\| \bar{Y}_{k\ell} \right\| \times \left\| V_k \right\| = \left\| \begin{matrix} I_1 \\ I_2 \\ \cdot \\ \cdot \\ I_n \end{matrix} \right\| = \left\| \begin{matrix} I_1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{matrix} \right\| + \left\| \begin{matrix} 0 \\ I_2 \\ \cdot \\ \cdot \\ 0 \end{matrix} \right\| + \cdots + \left\| \begin{matrix} 0 \\ 0 \\ \cdot \\ \cdot \\ I_n \end{matrix} \right\| \tag{6-1}$$

$$\left\| V_k \right\| = \left\| \bar{Y}_{k\ell} \right\|^{-1} \times \left[\left\| \begin{matrix} I_1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{matrix} \right\| + \left\| \begin{matrix} 0 \\ I_2 \\ \cdot \\ \cdot \\ 0 \end{matrix} \right\| + \cdots + \left\| \begin{matrix} 0 \\ 0 \\ \cdot \\ \cdot \\ I_n \end{matrix} \right\| \right] \tag{6-2}$$

This result may be stated as a theorem.

THEOREM III. In a class A network containing several independent sources the response caused by all the sources acting simultaneously equals the sum of the responses caused by each of the sources acting separately.

B. Thevenin's Theorem

Perhaps the most useful single theorem in steady state

network theory is Thevenin's Theorem. This will now be extended to transient class A four-terminal networks.

From equation 5-1a

(6-3) $I_2 = y_{21} V_1 + y_{22} V_2$

If the load admittance is Y_L,

(6-4) $I_2 = y_{21} V_1 - y_{22} \frac{I_2}{Y_L}$

(6-5) $I_2 = \dfrac{\frac{y_{21}}{y_{22}} V_1}{\frac{1}{y_{22}} + \frac{1}{Y_L}}$

Now if Y_L becomes zero, V_2 becomes the open circuit voltage of the network or V_{02}.

From 6-3

(6-6) $V_{02} = - \frac{y_{21}}{y_{22}} V_1$

Therefore

(6-7) $I_2 = \dfrac{-V_{02}}{\frac{1}{y_{22}} + \frac{1}{Y_L}}$

Equation 6-7 in words is Theorem IV.

THEOREM IV. The current flowing in the load of a class A four-terminal network is the negative of the open circuit voltage divided by the sum of the load impedance and the reciprocal of the short circuit admittance seen from the output terminal pair.

VII. APPLICATION

A. Theoretical

As an example the transient response of a shunt peaked wide band amplifier with voltage feedback will now be calculated. The wiring diagram and equivalent circuit are shown on figure 4. The tube is a pentode so the grid plate capacitance and the plate conductance are negligible. G_g^1 and G_2 are always made negligible when practicable. Interest is centered on such a short time interval after the start of the transient that C_k and C_c may be replaced by short circuits.

By inspection

$$(7\text{-}1) \qquad V_1'' - V_2' = V_1 + V_1' - V_2' = V_1 + V_2 \frac{G_2}{G_1 + G_2}$$

and

$$(7\text{-}2) \qquad \begin{Vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{Vmatrix} = \begin{Vmatrix} G_g & 0 \\ g_m & sC + \dfrac{g_m G_2}{G_1 + G_2} \end{Vmatrix}$$

Then from equation 5-20a

$$(7\text{-}3) \qquad A' = \frac{-g_m}{sC + \dfrac{g_m G_2}{G_1 + G_2} + \dfrac{\Gamma G}{sG + \Gamma}}$$

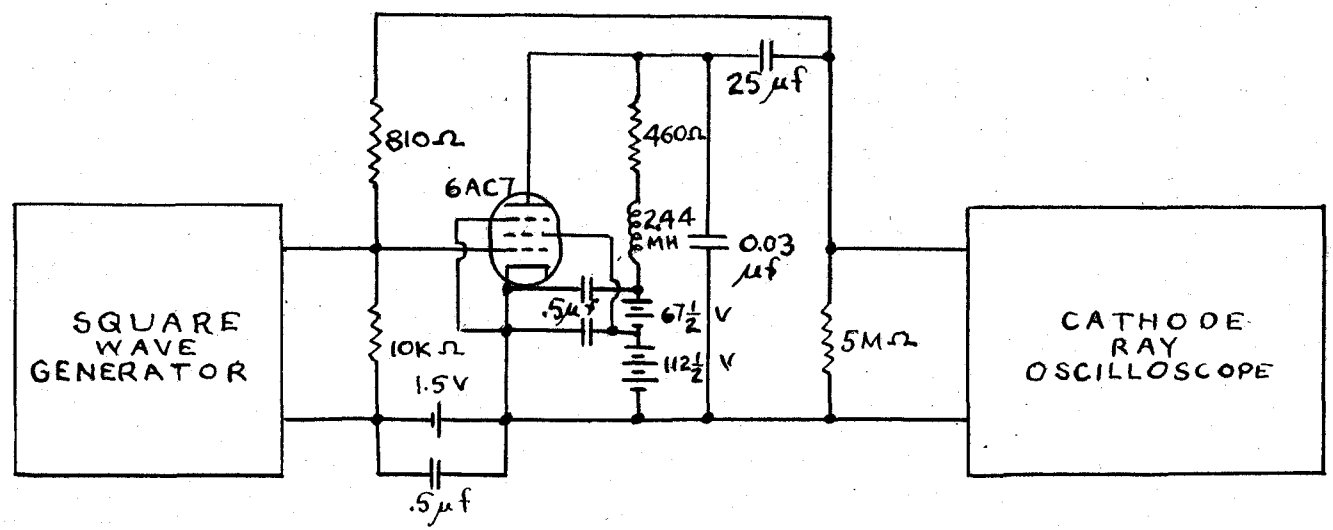


FIG. 6 CIRCUIT USED IN EXPERIMENT

GENERATOR: MEASUREMENTS CORP. MODEL 71 f = 7000 C.P.S.

OSCILLOSCOPE: DUMONT TYPE 241

For convenience let

$$(7\text{-}4)\quad \begin{cases} A_m' = -\dfrac{g_m}{G_T}, & Q = \dfrac{\omega_e G_T}{\Gamma} \\ \omega_e = \dfrac{G_T}{C} = 2\pi f_c, & \beta = \dfrac{G_{T_2}}{G_{T_1} + G_{T_2}} \end{cases}$$

Substituting these values and simplifying the equation,

$$(7\text{-}5)\quad \frac{A'}{A_m'} = \frac{\omega_e\left(s + \frac{\omega_e}{Q}\right)}{\left(s + \frac{\omega_e}{2Q} - \frac{A_m'\beta\omega_e}{2}\right)^2 - \left(\frac{\omega_e}{2Q} - \frac{A_m'\beta\omega_e}{2}\right)^2 + \frac{1 - A_m'\beta}{Q}\omega_e^2}$$

If the signal is a unit step voltage,

$$(7\text{-}6)\quad V_1(s) = \frac{1}{s}$$

Substituting this in equation 7-5

$$(7\text{-}7)\quad \frac{V_2(s)}{A_m'} = \frac{\omega_e\left(s + \frac{\omega_e}{Q}\right)}{s\left[\left(s + \frac{\omega_e}{2Q} - \frac{A_m'\beta\omega_e}{2}\right)^2 - \left(\frac{\omega_e}{2Q} - \frac{A_m'\beta\omega_e}{2}\right)^2 + \frac{1 - A_m'\beta}{Q}\omega_e^2\right]}$$

In reference (6) the authors came to the conclusion that of all the amplifiers they considered the best one was shunt peaked with $Q = 0.36$. They considered only the case of $\beta = 0$. Only this value of Q will be used and a comparison will be made of two values of β.

Equation 7-7 is of the form given in pair 1.305 (3, p. 343). This pair is given below

$$(7\text{-}8) \quad \mathcal{L}^{-1} \frac{s + a_0}{s\left[(s+\alpha)^2 + \gamma^2\right]} = \frac{a_0}{\beta_0^2} + \frac{\sqrt{(a_0-\alpha)^2 + \gamma^2}}{\gamma\beta_0} \varepsilon^{-\alpha t} \sin(\gamma t + \psi)$$

where

$$(7\text{-}9) \quad \begin{cases} \psi = \tan^{-1}\dfrac{\gamma}{a_0 - \alpha} - \tan^{-1}\dfrac{\gamma}{-\alpha} \\ \beta_0^2 = \gamma^2 + \alpha^2 \end{cases}$$

To make the response as nearly like the excitation as possible γ should be large to give a large initial rate of rise, and α should be large to damp out the overshoot. In addition as large a value of V_2 as possible should be obtained at $t = \infty$ in order to minimize the number of stages of amplification.

The latter value can be obtained by use of the Final Value Theorem of Laplace Transform Theory.

$$(7\text{-}10) \quad \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Applying this to 7-7

$$(7\text{-}11) \quad \lim_{t \to \infty} V_2(t) = \frac{A_m'}{1 - A_m'\beta}$$

Variations in 7-7 were studied while holding the output as given by 7-11 constant. A rather involved analysis showed that with this restriction no improvement in the response is possible. It will then also be impossible to increase the gain while holding the quality of the response constant. Because of the negative result obtained, the analysis will not be given here.

The other possibility was to accept some loss in gain in the hopes that a worthwhile improvement in the response would result. A value of β to cut the gain in half ($A_m'\beta = -1$) was tried. Q, of course, was set at 0.36. The value of $V_2(t)(1-A_m'\beta)/A_m'$ was calculated and is plotted on figure 5. The value of V_2/A_m' was calculated and plotted on figure 5 with $\beta = 0$ for comparison. The equations with numerical values substituted are given below.

$$(7\text{-}12) \quad \begin{cases} \beta = -\dfrac{1}{A_m'} \\ \dfrac{2V_2(t)}{A_m'} = 1 + 1.003\,\varepsilon^{-1.889\omega_e t}\sin(80.76^\circ\omega_e t - 85.5^\circ) \end{cases}$$

$$(7\text{-}13) \quad \begin{cases} \beta = 0 \\ \dfrac{V_2(t)}{A_m'} = 1 + 1.085\,\varepsilon^{-1.389\omega_e t}\sin(52.83^\circ\omega_e t - 112.8^\circ) \end{cases}$$

In plotting, $\omega_e t$ is used as a variable. This will be discussed later.

An amplifier with an ideal response would show an abrupt

rise to 1.0 at zero time with the output continuing at 1.0 thereafter. As far as the shape of the output is concerned, then, the feedback amplifier more nearly approaches the ideal.

It remains to consider whether the improvement in response is worth the loss in output for the unit step signal. (This loss in output will be present for all types of signal except an impulse which is a singular type.) It will be observed that the time of rise to 0.9 of the final value for the feedback amplifier is 55% of that for the amplifier without feedback. The overshoot of the feedback amplifier is 3.4% while that of the other is 1.3%. If the rise time of the original amplifier had been reduced to 55% of its former value, the overshoot would have been 19% (6). Suppose now that an amplifier with a gain of 20 is available. Reduce its gain to 10 by applying feedback. Now if two of these feedback amplifiers are connected in cascade, the resultant rise time is about the same as that of the single stage without feedback. On the other hand the gain is 100 with a slight increase in overshoot. Thus a significant increase in response is possible at the cost of an additional stage.

B. Experimental

Equations 7-12 and 7-13 are particularly useful for experimental work, because they are suited to the use of models. The fact that the transconductance of the tube does not

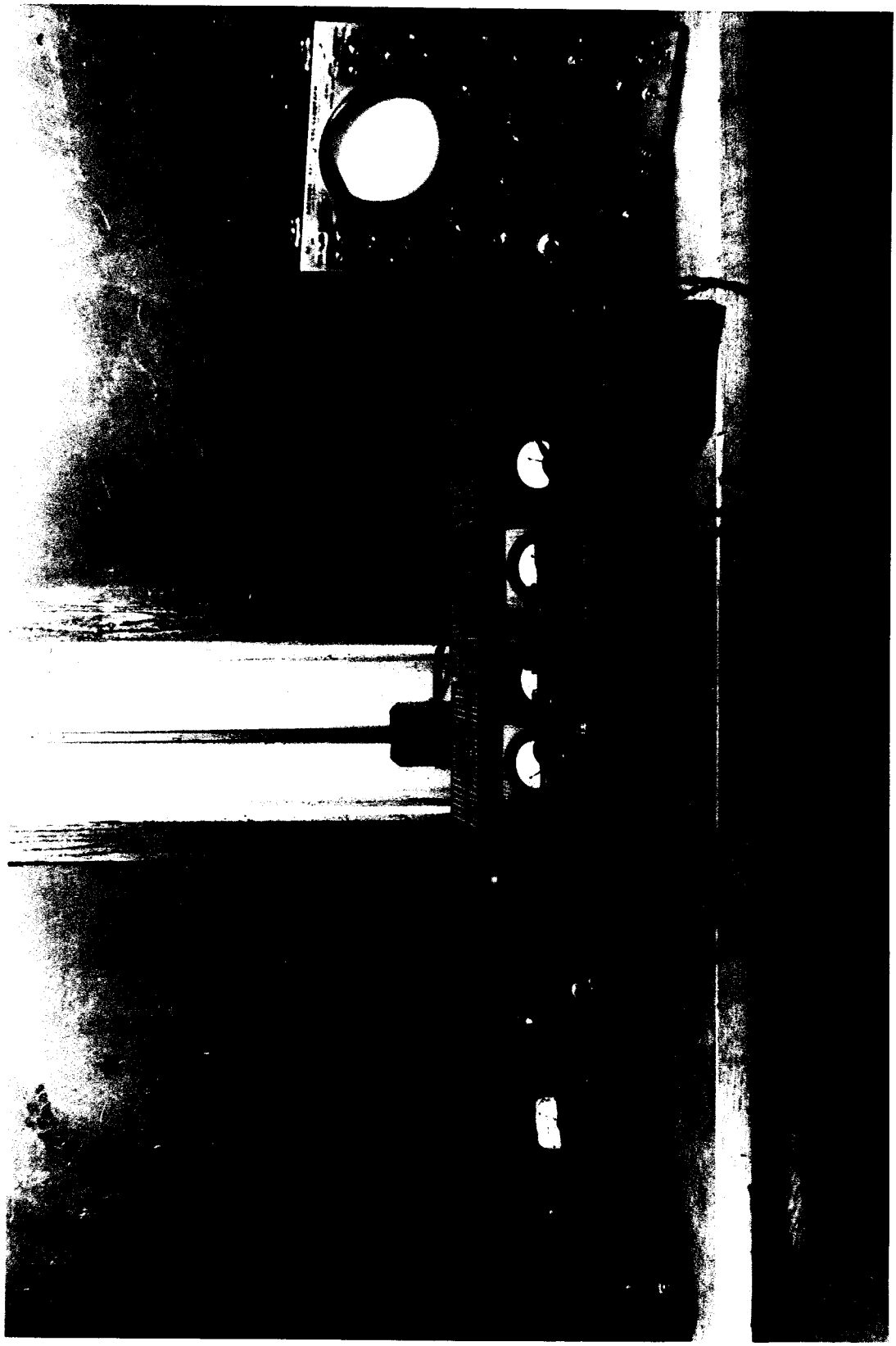


Fig. 7. Arrangement of Equipment in Laboratory

appear on the right hand side of the equation means that the results are applicable to any tube. Since t appears only when multiplied by ω_e means that the time scale can be altered as desired. This was done by using a large value of C. Thus the actual distributed capacitance of the circuit, which is hard to measure, was of negligible significance. In addition high frequency errors in the indicating instruments were avoided.

The circuit used is shown in figure 6. Figure 7 shows the laboratory setup. The components are (from left to right) square wave generator, feedback resistor, amplifier, load resistor, and oscilloscope. It will be noticed that the experimental circuit differs from figure 4 by the manner in which the feedback was applied. This was necessary because the grounds of the generator and the oscilloscope could not be separated without introducing considerable hum. The general theory of feedback amplifiers indicates that as long as the feedback network does not affect the gain of the amplifier before the feedback voltage is introduced (in the example studied A_m' should not be affected by the feedback network), the response of the amplifier will not be affected by the type of feedback network used. Unfortunately, because of the low internal impedance of the square wave generator, the feedback resistor had to be so small that there was some effect on the amplifier gain. It seemed to affect the value of ω_e

primarily thus changing the time scale of the response curve. This effect could not be calculated as the internal impedance of the generator was not known.

Oscillograms of the signal and output are shown in figures 8, 9, and 10. The square wave period was 1/7000 second; so the portion between the rise and subsequent drop was 1/14000 second. By calculation ω_e was 77000 radians per second, and Q was 0.36. The feedback resistor was adjusted until the output was one half that without feedback.

Points from the curves of figures 9 and 10 were replotted on figure 5. (The points were actually measured on the oscilloscope screen.) Without feedback the measured and calculated points checked very closely. With feedback the measured points gave a curve of proper shape but expanded time scale. It is this expanded time scale which was expected from the imperfect feedback circuit.

Fig. 8. Oscillogram of Signal Voltage

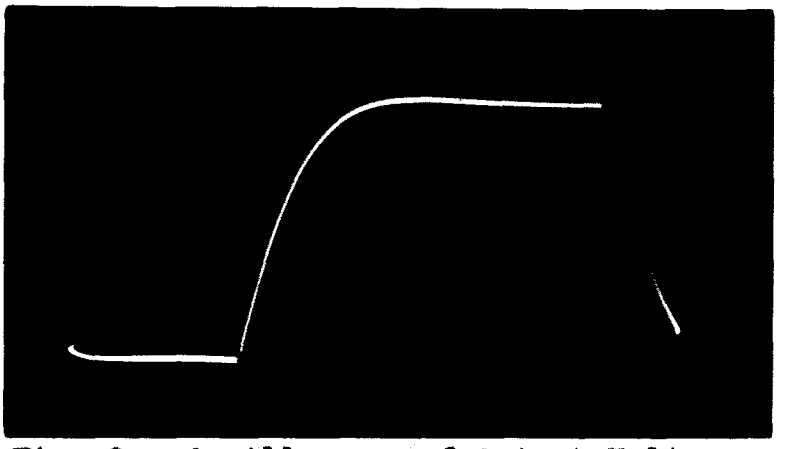


Fig. 9. Oscillogram of Output Voltage Without Feedback

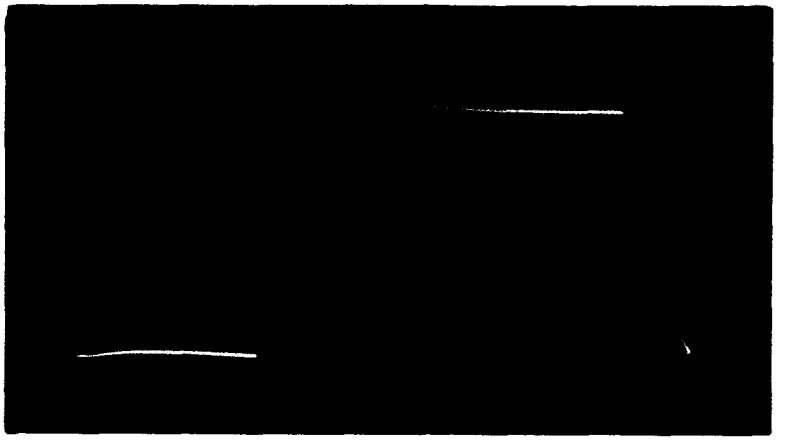


Fig. 10. Oscillogram of Output Voltage With Feedback

VIII. CONCLUSIONS

1. The a-c and transient responses of passive and class A networks are very closely related. The relation is given in Theorems I and II.

2. The treatment of four-terminal class A networks as a class can greatly simplify the study of such networks. The differences and similarities of such networks can be easily seen even if the internal connections differ appreciably.

3. The performance of wide band amplifiers may be greatly improved by the use of negative feedback.

IX. LITERATURE CITED

1. Bedford, A. V. and Fredendall, G. L. Transient response of multistage video-frequency amplifiers. Proc. Inst. Radio Engrs. 27:277-284. 1939.

2. Bôcher, M. Introduction to higher algebra. p. 60-79. New York, The Macmillan Company. 1912.

3. Gardner, M. F. and Barnes, J. L. Transients in linear systems. Vol. 1. New York, John Wiley and Sons, Inc. 1942.

4. Guillemin, E. A. Communication networks. Vol. 2. New York, John Wiley and Sons, Inc. 1935.

5. Jackson, W. The transient response of the triode valve equivalent network. Lond., Edin. and Dublin Phil. Mag. Ser. 7, 13:143-153. 1932.

6. Kallmann, H. E., Spencer, R. E. and Singer, C. F. Transient response. Proc. Inst. Radio Engrs. 33:169-195. 1945.

7. Miles, J. W. Junction analysis in vacuum-tube circuits. Proc. Inst. Radio Engrs. 32:617-620. 1944.

8. Mott, E. E. Indicial response of telephone receivers. Bell System Tech. Jour. 23:135-150. 1944.

9. Pipes, L. A. The matrix theory of four-terminal networks. Lond., Edin. and Dublin Phil. Mag. Ser. 7, 30:370-395. 1940.

X. ACKNOWLEDGMENTS

The writer wishes to express his appreciation to Professor M. S. Coover for his interest in and encouragement of the work reported here, to Professor W. L. Cassell and Dr. J. D. Ryder for many helpful discussions and suggestions, and to Dr. W. B. Boast and other members of the Electrical Engineering Department for their suggestions and interest.

XI. VITA

Author: Wilton Robert Abbott.

Father's name: Ernest Andrew Abbott.

Mother's maiden name: Audrey Keesling.

Date and place of birth: January 19, 1916, Campbell, California.

Elementary and secondary schools attended:

Horace Mann Grammar School, San Jose, California.

Hester Grammar School, San Jose, California.

Theodore Roosevelt Jr. High School, San Jose, California.

San Jose High School, San Jose, California.

Colleges and universities attended:

San Jose State Junior College, San Jose, California.

University of California, Berkeley, California.

Stanford University, Stanford, California.

Iowa State College, Ames, Iowa.

Degrees: Bachelor of Science, University of California, 1937.

Master of Science, Iowa State College, 1942.

Field of specialization: Electrical engineering, electronics, and communications.

Professor in charge of major work: Dr. J. D. Ryder.

Doctoral committee: Dr. J. D. Ryder

Dr. W. B. Boast

Dr. L. T. Earls

Dr. J. J. L. Hinrichsen

Dr. H. J. Stoever

XII. APPENDICES

A. Matrix Algebra

When systems of linear equations are to be manipulated, matrix algebra is of considerable use. The algebra was developed for this purpose, so the rules of manipulation were developed by analogy with the manipulations of systems of equations. See references (2) and (9). The operations and properties to be considered here are: equality, multiplication by a constant, multiplication of matrices, inversion, and addition.

A double subscript notation is often used to identify the elements of a matrix. The first subscript gives the number of the row in which the element is placed and the second the number of the column. As an example

$$(12\text{-}1) \qquad \left\| a_{k\ell} \right\|_{\substack{k=1,3 \\ \ell=1,2}} \qquad \left\| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} \right\|$$

The left hand side of this equation is a useful abbreviation. It indicates that the double subscript notation is to be used, that the rows are numbered from 1 to 3 and that the columns are numbered from 1 to 2. With a one column matrix the ℓ

subscript is ordinarily omitted.

Two matrices are equal if, and only if, all the elements of one are equal to the corresponding elements of the other. For instance,

$$\left\| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \right\| = \left\| \begin{matrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{matrix} \right\| \tag{12-2}$$

means that

$$\begin{cases} a_{11} = b_{11} \\ a_{12} = b_{12} \\ a_{21} = b_{21} \\ a_{22} = b_{22} \end{cases} \tag{12-3}$$

To multiply a matrix by a constant multiply each element of the matrix by the constant

$$k \left\| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \right\| = \left\| \begin{matrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{matrix} \right\| \tag{12-4}$$

Suppose that the product of two matrices is given by

$$\| a_{k\ell} \| \times \| b_{k\ell} \| = \| c_{k\ell} \| \quad , \quad k, \ell = 1, n \tag{12-5}$$

The rule is that

$$c_{k\ell} = a_{k1} b_{1\ell} + a_{k2} b_{2\ell} + \cdots + a_{kn} b_{n\ell} \tag{12-6}$$

Missing rows or columns may be filled out by adding rows or columns with zero elements. As an example,

$$\begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix} \times \begin{Vmatrix} b_{11} & 0 & 0 \\ b_{21} & 0 & 0 \\ b_{31} & 0 & 0 \end{Vmatrix} = \begin{Vmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & 0 & 0 \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & 0 & 0 \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & 0 & 0 \end{Vmatrix} \tag{12-7}$$

This equation would usually be written in the shorter form,

$$\begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix} \times \begin{Vmatrix} b_{11} \\ b_{21} \\ b_{31} \end{Vmatrix} = \begin{Vmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} \end{Vmatrix} \tag{12-7a}$$

Matrix multiplication is <u>not</u> commutative.

Finding the reciprocal of, or inverting, a matrix is possible only with a square matrix. The determinant of a matrix is the determinant having the same elements as the matrix. It will be denoted by D when no ambiguity results. The inverse of a matrix is found by replacing each element of the matrix by its minor in the corresponding determinant, interchanging rows and columns, and dividing the resulting matrix by D. For example

$$(12\text{-}8)\qquad \left\| \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \right\|^{-1} = \frac{1}{D} \left\| \begin{matrix} (a_{22}a_{33}-a_{23}a_{32}) & -(a_{12}a_{33}-a_{13}a_{32}) & (a_{12}a_{23}-a_{13}a_{22}) \\ -(a_{21}a_{33}-a_{23}a_{31}) & (a_{11}a_{33}-a_{13}a_{31}) & -(a_{11}a_{23}-a_{21}a_{13}) \\ (a_{21}a_{32}-a_{22}a_{31}) & -(a_{11}a_{32}-a_{12}a_{31}) & (a_{11}a_{22}-a_{21}a_{12}) \end{matrix} \right\|$$

where

$$(12\text{-}9)\qquad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

To add two matrices, add the corresponding elements of the two.

$$(12\text{-}10)\qquad \| a_{k\ell} \| + \| b_{k\ell} \| = \| a_{k\ell} + b_{k\ell} \|$$

To subtract two matrices multiply the subtrahend by -1 and add.

To illustrate the application of matrix algebra to the manipulation of systems of linear equations consider the following equation

$$(12\text{-}11)\qquad \left\| \begin{matrix} c_1 \\ c_2 \end{matrix} \right\| = \left\| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \right\| \times \left\| \begin{matrix} b_1 \\ b_2 \end{matrix} \right\| = \left\| \begin{matrix} a_{11}b_1 + a_{12}b_2 \\ a_{21}b_1 + a_{22}b_2 \end{matrix} \right\|$$

This is equivalent to

$$(12\text{-}12) \qquad \begin{cases} c_1 = a_{11} b_1 + a_{12} b_2 \\ c_2 = a_{21} b_1 + a_{22} b_2 \end{cases}$$

Thus the matrix equation 12-11 is equivalent to the system 12-12. Suppose it is desired to solve for the b's in terms of the c's. Multiply both sides of the equation by the inverse of the matrix $\|a\|$

$$(12\text{-}13) \qquad \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{Vmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{Vmatrix}$$

$$(12\text{-}14) \qquad \begin{Vmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{Vmatrix} \times \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} = \begin{Vmatrix} a_{22}a_{11} - a_{12}a_{21} & a_{22}a_{12} - a_{12}a_{22} \\ -a_{21}a_{11} + a_{11}a_{21} & -a_{21}a_{12} + a_{11}a_{22} \end{Vmatrix}$$

The right hand side of 12-14 further reduces to

$$(12\text{-}15) \qquad \begin{Vmatrix} a_{22}a_{11} - a_{12}a_{21} & 0 \\ 0 & a_{22}a_{11} - a_{12}a_{21} \end{Vmatrix} = (a_{22}a_{11} - a_{12}a_{21}) \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$$

Then 12-11 multiplied by the inverse of the a matrix becomes

$$(12\text{-}16) \qquad \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{Vmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{Vmatrix} \times \begin{Vmatrix} c_1 \\ c_2 \end{Vmatrix} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \times \begin{Vmatrix} b_1 \\ b_2 \end{Vmatrix} = \begin{Vmatrix} b_1 \\ b_2 \end{Vmatrix}$$

Thus the equation is solved for the b's. Ordinarily it would not be necessary to go through all these steps as the product of a matrix and its inverse is evidently unity.

The above operations are basic and will serve to solve any problem likely to be met with by the circuit analyst.

B. Circuit Analysis

In most cases vacuum tube circuits are more easily analyzed in terms of node voltages than loop currents. While this method is described in many places (7), it is not so widely known that it will not be worth while to review it here briefly.

A node is a point in a circuit connected to two or more circuit elements. Kirchoff's second law says, "The sum of the currents leaving a node at any instant is zero". It is this law which is basic in the node voltage method of analysis.

To use this method first number all nodes beginning with zero. The zero node should be ground or whatever point is

most convenient as a reference. If the circuit contains a single vacuum tube, it would be well to consider using the cathode of the tube as reference. There will be an equation for every node but zero.

If there are, say, four elements connecting to node 8, there will be four terms in the equation. The other end of each of the elements may terminate on any node. In the most general circuit there would be a resistance, capacitance, inductance, and current source from each node to each other node. Voltage sources which may be present merely set the potential difference between two nodes; so they are not considered in the same light as other circuit elements. In the example suppose that one of the four elements is a resistance connected to node 7, one an inductance connected to node 6, one a capacitance connected to node 5 and one a current source connected to node 4. If v_n represents the potential of node n with respect to node zero, then the total current leaving node 8 is

$$\text{(12-17)} \quad \frac{1}{R_{87}}(v_8 - v_7) + \frac{1}{L_{86}}\int (v_8 - v_6)\,dt + C_{85}\frac{d}{dt}(v_8 - v_5) + i_{84} = 0$$

It is convenient to let

$$\text{(12-18)} \quad G = \frac{1}{R}, \quad \Gamma = \frac{1}{L}$$

Transforming 12-17 and collecting terms,

$$(12\text{-}19) \quad V_8(s)\left[G_{87} + \frac{\Gamma_{86}}{s} + sC_{85}\right] - G_{87}V_7(s) - \frac{\Gamma_{86}}{s}V_6(s) - sC_{85}V_5(s) = -I_{84}(s)$$

Since the developments of this thesis apply to circuits initially at rest, initial charges and currents were omitted from 12-19. The problem is more complicated if there is mutual inductance in the circuit. See reference (2, p. 40). That situation will not be discussed further here as it does not enter into any of the examples given.

If

$$(12\text{-}20) \quad Y_{jk}(s) = G_{jk} + \frac{\Gamma_{jk}}{s} + sC_{jk}$$

Where G_{jj} is the sum of all G's connected to node j, Γ_{jj} is the sum of all Γ's connected to node j, C_{jj} is the sum of all C's connected to node j, $-G_{jk}$ is the conductance from j directly to k, $-\Gamma_{jk}$ is the reciprocal inductance from j directly to k and $-C_{jk}$ is the capacitance directly from j to k. If also $I_j(s)$ is the transform of the total current entering node j from current sources, then the following matrix equation for the network follows. See reference (9).

(12-21) $$\left\| Y_{jk}(s) \right\| \times \left\| V_j(s) \right\| = \left[\left\| G_{jk} \right\| + \frac{1}{s} \left\| \Gamma_{jk} \right\| + s \left\| C_{jk} \right\| \right] \times \left\| V_j(s) \right\| = \left\| I_j(s) \right\|$$

$$j, k = 1, n$$

If a voltage source E is connected, say, from node 3 to node 4 with the positive to node 3, the following relation obviously holds.

(12-22) $$V_3 - V_4 = E$$

Using this relation V_3 or V_4 can be eliminated from the system 12-21. In most cases it will be more satisfactory to make an exchange of sources as in reference (3, p. 43) before commencing the circuit analysis. Each change of a voltage source into a current source reduces the number of nodes by one.

The choice as to whether to use the node voltage or loop current attack in a particular problem rests on which leads to the smaller number of unknowns.

In section V it is shown that input and transfer short circuit admittances are of considerable importance. Short circuiting the output reduces the number of nodes by at least one and may reduce the number of loops by a considerably larger number. It should not be forgotten that this may swing

the balance in favor of loop analysis. Of course, Thevenin's Theorem and other aids to computation should always be considered before starting a more general analysis. Theorems I and II, when applicable, will be of considerable aid.

C. Symbols

The following list of symbols covers only those used widely in the thesis. Definitions of other symbols will be found in the text material as the symbols occur. Subscript definitions will also be found in the text.

A'	ratio of output to input voltage of a four-terminal network.
C	capacitance.
D	determinant of a matrix.
F	Laplace Transform of f.
f	a function of time.
G	conductance.
g_m	transconductance of a vacuum tube.
I	transform of a current.
i	instantaneous current.
j	$\sqrt{-1}$
$\mathcal{L}$	operator; read, "Laplace Transform of".
r_p	plate resistance of a vacuum tube.

s the variable of the Laplace Transform.

t time.

V transform of a voltage.

v instantaneous voltage.

Y,y admittance.

Z,z impedance.

Γ reciprocal of inductance.

ω angular velocity.

μ amplification factor of a vacuum tube.

ε base of natural logarithms.

D. Tabulated Data

$\omega_e t$	Output Voltage (without feedback)	Output Voltage (with feedback)
0.6	0.55	0.7
1.2	0.86	0.975
1.8	0.99	1.04
2.4	1.02	1.025
3.0	1.02	1.005
3.6	1.005	1.00
4.2	1.005	1.00
4.8	1.00	1.00

Data plotted on figure 5.